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COMMENT

On the Fokker–Planck equation with force term

T W Marshall and E J Watson

Department of Mathematics, University of Manchester, Manchester M13 9PL, UK

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Abstract. It has been suggested that the half-range Sturm–Liouville problem, produced in the study of the one-dimensional Fokker–Planck equation with constant force term, can be replaced by two definite Sturm–Liouville problems. We show that this appears not to be the case. We also draw attention to some corrections to our earlier papers.

We are interested to note that, in his recent letter with the above title, Protopopescu (1987) has established the half-range completeness property for the stationary one-dimensional Fokker–Planck equation that we used in a study (Marshall and Watson 1987) of the albedo and Milne problems. Protopopescu went on to suggest that the half-range Sturm–Liouville problem thus produced could be replaced by two definite Sturm–Liouville problems. However, it appears that this does not lead to the required solution of the partial differential equation

$$\frac{\partial^2 P}{\partial u^2}+(u+2\alpha)\frac{\partial P}{\partial u}+P-u\frac{\partial P}{\partial x}=0 \tag{1}$$

with

$$P\to 0 \qquad \text{as} \qquad u\to\pm\infty \tag{2}$$

$$P(0,u)=g(u) \qquad u>0 \tag{3}$$

$$P\to 0 \qquad \text{as} \qquad x\to+\infty. \tag{4}$$

Separation of the variables in (1), in the form

$$P(x,u)=\exp[-(\lambda+\alpha)x-\tfrac{1}{4}u^2-\alpha u]\varphi(u) \tag{5}$$

gives

$$\varphi''(u)-(\tfrac{1}{4}u^2-\lambda u+\alpha^2-\tfrac{1}{2})\varphi(u)=0. \tag{6}$$

The boundary conditions (2) lead to

$$\varphi(u)\to 0 \qquad \text{as} \qquad u\to\pm\infty \tag{7}$$

so that we took

$$\varphi(u)=D_n(u-2\lambda) \qquad n=0,1,2,\ldots \tag{8}$$

where

$$\lambda^2=n+\alpha^2 \tag{9}$$

to obtain the half-range expansion problem

$$\sum_{n=0}^{\infty} C_n D_n[u-2(n+\alpha^2)^{1/2}] = g(u)\exp(\tfrac{1}{4}u^2+\alpha u) \qquad u>0. \tag{10}$$

Protopopescu's suggestion is that a boundary condition of the form

$$\varphi(0)\cos\beta - \varphi'(0)\sin\beta = 0 \qquad 0 \leqslant \beta < \pi \tag{11}$$

should be imposed, in order to obtain a definite Sturm–Liouville problem in $0<u<\infty$. The case $\beta=0$ was considered by Dita (1985). Thus equation (11) has to be applied to the solution

$$\varphi_\lambda(u) = D_{\lambda^2-\alpha^2}(u-2\lambda) \tag{12}$$

of (6) that vanishes as $u\to+\infty$. This would lead to a function

$$P(x,u) = \exp(-\tfrac{1}{4}u^2-\alpha u-\alpha x)\sum_{n=0}^{\infty} C_n \exp(-\lambda_n x)\varphi_{\lambda_n}(u) \tag{13}$$

that satisfies equation (1) with $P\to 0$ as $u\to+\infty$ and

$$P(x,0)(\cos\beta-\alpha\sin\beta) - \frac{\partial P}{\partial u}(x,0)\sin\beta = 0 \tag{14}$$

for all $x>0$. For the required solution of (1)–(4) the ratio $(\partial P/\partial u)/P$ at $u=0$ is not constant, so that it would be necessary to decompose P into two functions, for instance as

$$P(x,u) = P_1(x,u)+P_2(x,u) \tag{15}$$

where

$$P_1(x,0) = \frac{\partial P_2}{\partial u}(x,0) = 0 \qquad \text{for} \qquad x>0. \tag{16}$$

Each of the functions P_1 and P_2 would have an expansion of type (13), but the eigenvalues λ_n would be different. When $u\to-\infty$, the terms of these expansions grow like $\exp(\lambda_n^{1/2}|u|^{3/2})$, so that although the series will converge it would be difficult to use them in order to satisfy the condition $P\to 0$ as $u\to-\infty$. This means that another representation of $P(x,u)$ would be required in $u<0$. Equation (11) would then have to be applied to the solution

$$\varphi_\lambda(u) = D_{\lambda^2-\alpha^2}(2\lambda-u) \tag{17}$$

that vanishes as $u\to-\infty$, and in this case the eigenvalues $\lambda_n\to-\infty$ as $n\to\infty$, so that for $u<0$ the series corresponding to (13) would be applicable in $x<0$ only. Thus we conclude that it is impracticable to apply any condition of the form (11). We believe that the use of the Wiener–Hopf decomposition, or of an analogous projection operation, is essential to the solution of problems such as (1)–(4).

We wish to call attention to the following corrections to Marshall and Watson (1985):
p 3534, after (2.14) should be $\Phi(y, v; t) \geqslant 0$
p 3539, after (3.41) should be $L_2(s,\alpha) = O(|s|^{-7/6})$
p 3545, (4.23) $\bar{\Phi}(y;p) = \exp[(\alpha-(\alpha^2+p)^{1/2})y]$
p 3551, (A1.31) should be $q^{q^2-\tau+2/3}$.

Corrections to Duck *et al* (1986):

p 3550, *l*13, $G(0, \Delta\xi)$ should be $F(\xi, 0)$

p 3550, *l*14, $F(0, \xi)$ should be $G(\Delta\xi, 0)$.

Corrections to Marshall and Watson (1987):

p 1346, (1.15) $q_n = (n+\alpha^2)^{1/2}$

p 1347, (2.4) should be $(8\pi)^{-1/2}$

p 1349, the sentence after (2.22) should say that 'the Uhlenbeck-Ornstein process is recurrent for $\alpha \geqslant 0$'

p 1349, (3.5) $n^{1/2n-1}$ should be $n^{n/2-1}$

p 1350, (3.7) Br should be B_r

table 1 heading, should be $x - \zeta\ (\frac{1}{2}) - n^M(x)$

p 1352, (4.14) should be $\psi(0, v) = -\frac{1}{2}\alpha^{-1}v$

p 1353, (4.18) should be $(1+\frac{1}{4}\alpha^{-2})$

p 1354, (A8) should be $q-(n+\tau)^{1/2}$.

References

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